Dyon death eaters

Anindya Mukherjee, Sunil Mukhi and Rahul Nigam

Tata Institute of Fundamental Research,
Homi Bhabha Rd, Mumbai 400 005, India
E-mail: `anindya_m@theory.tifr.res.in`, `mukhi@tifr.res.in`,
`rahulnig@theory.tifr.res.in`

ABSTRACT: We study general two-body decays of primitive and non-primitive $\frac{1}{4}$-BPS dyons in four-dimensional type IIB string compactifications. We find a "master equation" for marginal stability that generalises the curve found by Sen for $\frac{1}{2}$-BPS decay, and analyse this equation in a variety of cases including decays to $\frac{1}{4}$-BPS products. For $\frac{1}{2}$-BPS decays, an interesting and useful relation is exhibited between walls of marginal stability and the mathematics of Farey sequences and Ford circles. We exhibit an example in which two curves of marginal stability intersect in the interior of moduli space.

KEYWORDS: Superstrings and Heterotic Strings, Black Holes in String Theory.

JHEP10(2007)037

Contents

1. Introduction

In the last couple of years there has been renewed interest in the properties of dyonic black holes in four dimensions, particularly those associated to $\mathcal{N} = 4$ compactifications (type II strings on $K3 \times T^2$ or heterotic/type I strings on T^6, as well as supersymmetry-preserving orbifolds of these systems) [1–14]. A key advance has been a better understanding of a classic degeneracy formula due to Dijkgraaf, Verlinde and Verlinde [15]. Among other things, the generalisation of this formula to CHL orbifolds and the origin of a genus-2 modular form have been illuminated in many of these works.

Recent work has focused on the issue of marginal stability of these dyons. Curves of marginal stability for specific decays have been obtained [9], the impact of such decays on the degeneracy formula has been studied [9, 10, 13] and the decays across such walls have been identified with the disappearance of two-centred black holes from the spectrum [12, 13], following previous ideas in the $\mathcal{N} = 2$ context [16]. A formula has been proposed in [13] to count the "immortal" dyons which exist everywhere. And very recently, Sen has considered the case of primitive dyons decaying into $\frac{1}{4}$-BPS states [14] and demonstrated that this takes place only on surfaces of codimension 2 in moduli space.

Clearly there is much more to be learned about this system. Among various interesting questions is a complete understanding of the possible marginal decays of $\frac{1}{4}$-BPS dyons, the

impact of such decays on the degeneracy counting function, the role of multi-centred black holes in the decays, and the relevance of "non-primitive" dyons (which are related to Riemann surfaces of genus $g > 2$) to the counting problem.

In the present work we take a step towards resolving the first problem. We consider the most general decay of a $\frac{1}{4}$-BPS dyon into two decay products, each one of which can be either $\frac{1}{2}$- or $\frac{1}{4}$-BPS. We find a necessary condition for marginal two-body decays and study the resulting equation in a variety of cases. It turns out that some solutions of our equation are "spurious" in the sense that they describe an inverse decay process rather than the forward decay. This puts constraints on the possible decay products which are identical to those found in [14]. We are also able to reproduce some of the results in refs. [9] as a special case, as well as generalise them to the case of "non-primitive" dyons. On the way we will see that a known mathematical construction, that of Farey sequences and Ford circles, bears a remarkably close relation to the circles of marginal stability in ref. [9] and helps us understand the properties of these circles.

2. The system

We consider type IIB string theory compactified on $K3 \times T^2$. The resulting four-dimensional system has 28 $U(1)$ gauge fields and their electric-magnetic duals. Therefore we can have dyons of charge $(\vec{Q}, \vec{P})$ where the first entry is a 28-component vector denoting electric charge under these gauge fields and the second denotes the magnetic charge. The dyons will be $\frac{1}{2}$-BPS if the vectors $\vec{Q}, \vec{P}$ are parallel, and $\frac{1}{4}$-BPS otherwise.

Note that a modular transformation of the 2-torus T^2 that changes its modular parameter as:

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{2.1}$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, Z) \tag{2.2}$$

sends the dyon charges to:

$$\begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix} \to \begin{pmatrix} a\vec{Q} + b\vec{P} \\ c\vec{Q} + d\vec{P} \end{pmatrix} \tag{2.3}$$

We are interested in the marginal decays of these $\frac{1}{4}$-BPS dyons. The stability or otherwise is dictated by the charges carried by the dyons as well as the values of the moduli of $K3 \times T^2$. These are encoded as follows. Define the matrix:

$$L \equiv \mathrm{diag}(1^6; (-1)^{22}) \tag{2.4}$$

In 4 dimensions there are, first of all, 132 moduli that can be assembled into a matrix M that is symmetric and orthogonal with respect to the L metric:

$$M^T = M, \quad M^T L M = L \tag{2.5}$$

The relevant inner product for an electric charge vector, which we will call Q^2 or $\vec{Q} \cdot \vec{Q}$, is:[1]

$$Q^2 \equiv \vec{Q}^T (M + L) \vec{Q} \tag{2.6}$$

Correspondingly we have:

$$\begin{aligned} P^2 &\equiv \vec{P}^T (M + L) \vec{P} \\ P \cdot Q &\equiv \vec{P}^T (M + L) \vec{Q} \end{aligned} \tag{2.7}$$

We will also make use of the quantities $\vec{Q}_R, \vec{P}_R$ defined such that

$$Q_R^2 \equiv \vec{Q}_R^T \vec{Q}_R = \vec{Q}^T (M + L) \vec{Q} \tag{2.8}$$

and similarly for the other inner products (for details see for example [13, 14]).

In addition to the moduli appearing in M, there is the modular parameter of the 4-5 torus:

$$\tau = \tau_1 + i\tau_2 \tag{2.9}$$

The BPS mass formula for general $\frac{1}{4}$-BPS dyons is [17, 18]:

$$M_{\text{BPS}}(\vec{Q}, \vec{P})^2 = \frac{1}{\sqrt{\tau_2}} (\vec{Q} - \bar{\tau}\vec{P}) \cdot (\vec{Q} - \tau\vec{P}) + 2\sqrt{\tau_2}\sqrt{\Delta(\vec{Q}, \vec{P})} \tag{2.10}$$

where

$$\Delta(\vec{Q}, \vec{P}) \equiv Q^2 P^2 - (P \cdot Q)^2 \tag{2.11}$$

Before going on, it is useful to transform the dyon charges to bring them into a standard form. Consider the electric and magnetic charge vectors $\vec{Q}, \vec{P}$ of the dyon and define [10]:

$$I(\vec{Q}, \vec{P}) \equiv gcd(\vec{Q} \wedge \vec{P}) = gcd(Q^i P^j - Q^j P^i), \text{ all } i, j \tag{2.12}$$

If for a given dyon we find that $I(\vec{Q}, \vec{P}) > 1$, we first perform an SL(2, Z) transformation as in eq. (2.3). Using some properties of finitely generated algebras (see for example ref. [19], section I, section 8), we can always find such a transformation[2] that yields new dyon charges of the form $(m\vec{Q}', n\vec{P}')$ for some positive integers m, n and some new vectors $\vec{Q}', \vec{P}'$ such that $I(\vec{Q}', \vec{P}') = 1$. Under this transformation $I(\vec{Q}, \vec{P})$ remains invariant, so m, n must be such that $I(\vec{Q}, \vec{P}) = mn$. If the m, n so obtained are not co-prime then the dyon with those m, n will be marginally unstable at all points of moduli space. This does not mean a bound state does not exist, but that determining its existence is more subtle and requires actually quantising the system. Therefore we will restrict ourselves to the case where m, n are co-prime.

To summarise, in what follows we assume that our dyons have charge vectors $(m\vec{Q}, n\vec{P})$ with co-prime m, n and with $I(\vec{Q}, \vec{P}) = 1$. The special case $(m, n) = (1, 1)$ will be called a *primitive* dyon.

---

[1]Because our focus is on microstates, our inner products are always defined with respect to the moduli at infinity, so this notation should not cause confusion.

[2]We are grateful to Nitin Nitsure for helpful discussions on this point.

3. Decays into a pair of dyons

We can now examine the decay of a $\frac{1}{4}$-BPS dyons into two other dyons. From charge conservation, the most general decay is of the form:

$$\begin{pmatrix} m\vec{Q} \\ n\vec{P} \end{pmatrix} \rightarrow \begin{pmatrix} \vec{Q}_1 \\ \vec{P}_1 \end{pmatrix} + \begin{pmatrix} m\vec{Q} - \vec{Q}_1 \\ n\vec{P} - \vec{P}_1 \end{pmatrix} \tag{3.1}$$

where $\vec{Q}_1, \vec{P}_1$ are arbitrary vectors in the $(6, 22)$-dimensional integral charge lattice.

From the study of BPS string junctions and networks [20 – 22], we know that the decay products can be mutually BPS with each other and with the initial state only if the corresponding charges all lie in a plane rather than being generic 28-dimensional vectors as above. However, the properties of the networks are determined in the present context not by the charge vectors $\vec{Q}, \vec{P}$ but by their projections $\vec{Q}_R, \vec{P}_R$. Indeed it is only the latter which appear in the BPS mass formula eq. (2.10) that we will be using. Therefore the BPS condition requires that the R projections of the final-state charges are in the same plane as those of the initial-state charges. This happens automatically in some cases, while in others it requires adjusting the moduli in M to make this happen.

It follows that we must have the relation:

$$\begin{pmatrix} m\vec{Q}_R \\ n\vec{P}_R \end{pmatrix} \rightarrow \begin{pmatrix} m_1\vec{Q}_R + r_1\vec{P}_R \\ s_1\vec{Q}_R + n_1\vec{P}_R \end{pmatrix} + \begin{pmatrix} m_2\vec{Q}_R + r_2\vec{P}_R \\ s_2\vec{Q}_R + n_2\vec{P}_R \end{pmatrix} \tag{3.2}$$

where the coefficients m_i, n_i, r_i, s_i satisfy:

$$m_1 + m_2 = m, \quad n_1 + n_2 = n, \quad r_1 + r_2 = s_1 + s_2 = 0 \tag{3.3}$$

We cannot, however, assume that these coefficients are integers since the above equation refers not to the original vectors in the integral lattice but to their projections to the $\vec{Q}_R, \vec{P}_R$ plane.

Without any additional conditions on these coefficients the decay products will both be $\frac{1}{4}$-BPS dyons. It is possible to have one or both of them be $\frac{1}{2}$-BPS by suitably constraining the integers, as we will see shortly.

If M, M_1, M_2 denote the BPS masses of the initial state and the two decay products (for simplicity we henceforth drop the subscript BPS), we can use eqs. (2.10) and (3.2) to evaluate the condition on the moduli imposed by the marginality condition $M = M_1 + M_2$. Because of the square root in eq. (2.10), this is most easily done by computing a combination of squared masses that vanishes when the marginality condition is satisfied.

First, define the angles θ and θ_{12} by:

$$\theta = \tan^{-1}\frac{\tau_2}{\tau_1}$$
$$\theta_{QP} = \cos^{-1}\frac{Q_R \cdot P_R}{Q_R P_R} \tag{3.4}$$

where $Q_R \equiv |\vec{Q}_R|, P_R \equiv |\vec{P}_R|$. Geometrically, θ is the opening angle of the torus while θ_{QP} is the angle between the projected electric and magnetic charge vectors (which coincides

with the angle appearing in the string junction description of the dyon). We also define a "cross-product" between the integers m_1, n_1, m_2, n_2 as:

$$m \wedge n = m_1 n_2 - m_2 n_1 \tag{3.5}$$

Let us now find the condition that, at some point(s) in moduli space, the decay eq. (3.2) becomes marginal: $M = M_1 + M_2$. The BPS formula eq. (2.10) involves a square root on the r.h.s. and another square root to extract M from M^2. The simplest square-root-free expression that vanishes when the marginality condition is satisfied is the combination:

$$
\begin{aligned}
M^4 &+ M_1^4 + M_2^4 - 2(M^2 M_1^2 + M^2 M_2^2 + M_1^2 M_2^2) \\
&= (M - M_1 - M_2)(M + M_1 + M_2)(M - M_1 + M_2)(M + M_1 - M_2)
\end{aligned} \tag{3.6}
$$

Now we require this expression to vanish. However, subsequently we must check that on the vanishing curve, it is really the first factor of the r.h.s. of eq. (3.6) that vanishes rather than any of the other factors. Notice that the second factor never vanishes (since all the M's are positive), while vanishing of the third or fourth factor corresponds to the inverse decays $M_1 = M + M_2$ and $M_2 = M + M_1$. When we turn to a detailed analysis of marginal decay processes, it will be necessary to rule out these inverse decays before concluding that we are dealing with the correct decay mode. Only in the case where both the final products are $\frac{1}{2}$-BPS, this check becomes unnecessary because the reverse process is forbidden: a $\frac{1}{2}$-BPS state cannot decay into a $\frac{1}{4}$-BPS state.

Now we use the BPS mass formula eq. (2.10), the formula for the decay process eq. (3.2), and and the definitions of the angles in eq. (3.4), to find after a tedious calculation that:

$$
\begin{aligned}
M^4 + M_1^4 + M_2^4 - 2(M^2 M_1^2 + M^2 M_2^2 + M_1^2 M_2^2) = -4\tau_2^2 \Bigg[& Q_R P_R \frac{\sin(\theta + \theta_{QP})}{\sin\theta} \, m \wedge n \\
+ r_1 P_R \left(m Q_R \frac{\sin\theta_{QP}}{|\tau|\sin\theta} + n P_R \right) &- s_1 Q_R \left(n P_R \frac{|\tau|\sin\theta_{QP}}{\sin\theta} + m Q_R \right) \Bigg]^2
\end{aligned} \tag{3.7}
$$

Vanishing of the r.h.s. is a necessary condition for marginal stability.

This condition can be usefully rewritten by eliminating the angles θ, θ_{QP} and reverting to τ_1, τ_2 coordinates for the modular parameter of the torus. It is convenient to introduce the following quantity depending on charges of the initial and final states as well as the moduli:

$$E \equiv \frac{1}{\sqrt{\Delta}} \left(m s_1 Q_R^2 - n r_1 P_R^2 - (m \wedge n) Q_R \cdot P_R \right) \tag{3.8}$$

Then we find that the equation for marginal stability is:

$$\left(\tau_1 - \frac{m \wedge n}{2 n s_1} \right)^2 + \left(\tau_2 + \frac{E}{2 n s_1} \right)^2 = \frac{1}{4 n^2 s_1^2} \left((m \wedge n)^2 + 4 m n r_1 s_1 + E^2 \right) \tag{3.9}$$

This is the "master equation" governing all two-body decays of $\frac{1}{4}$-BPS states in this theory. However we will need careful analysis to see when the equation does actually describe such a decay and what type of decay it describes.

Note first of all that the equation is invariant under the transformation:

$$r_1 \to r_2 = -r_1, \quad s_1 \to s_2 = -s_1, \quad m_1 \to m_2 = m - m_1, \quad n_1 \to n_2 = n - n_1 \quad (3.10)$$

under which $m \wedge n$ and E both change sign. This corresponds to interchange of the two decay products.

If the r.h.s. of eq. (3.9) can be shown to be positive definite, this will be a circle in the torus moduli space with centre at:

$$(\tau_1, \tau_2) = \left(\frac{m \wedge n}{2ns_1}, -\frac{E}{2ns_1} \right) \quad (3.11)$$

and radius

$$\frac{1}{2ns_1}\sqrt{(m \wedge n)^2 + 4mnr_1s_1 + E^2} \quad (3.12)$$

Because there is no restriction on the signs of r, s, it may appear that the r.h.s. of eq. (3.9) is not positive definite. However, after a little computation we are able to rewrite it as:

$$(m \wedge n)^2 + 4mnr_1s_1 + E^2 = \frac{1}{\Delta}\Big(\left[(m \wedge n)Q_R P_R - (ms_1 Q_R^2 - nr_1 P_R^2)\cos\theta_{QP}\right]^2$$
$$+ (ms_1 Q_R^2 + nr_1 P_R^2)^2 \sin^2\theta_{QP}\Big) \quad (3.13)$$

which is a sum of squares. Therefore the equation does indeed describe a nontrivial circle in every case.

The next step is to check whether this circle intersects the upper half-plane. There are two cases. If $\frac{E}{s_1} > 0$ then the centre of the circle is in the lower half plane. The circle will then intersect the upper half plane only if it intersects the real axis, which happens if:

$$(m \wedge n)^2 + 4mnr_1s_1 > 0 \quad (3.14)$$

It is easy to see that:

$$(m \wedge n)^2 + 4mnr_1s_1 = \operatorname{tr}\mathbf{F}^2 - 2\det\mathbf{F} \quad (3.15)$$

where

$$\mathbf{F} = \begin{pmatrix} nm_1 & nr_1 \\ ms_1 & mn_1 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix} \quad (3.16)$$

Now if α_1, α_2 are the eigenvalues of $\mathbf{F}$ then:

$$\operatorname{tr}\mathbf{F}^2 - 2\det\mathbf{F} = (\alpha_1 - \alpha_2)^2 \quad (3.17)$$

This is positive if α_1, α_2 are both real, and negative if they are complex conjugate pairs. Therefore when $\frac{E}{s_1}$ is positive, only decays for which the eigenvalues of $\mathbf{F}$ are real can produce genuine curves of marginal stability in the upper half plane of τ-space.

If $\frac{E}{s_1} < 0$ then the circle has its centre in the upper half plane, and therefore always has a finite region in the upper half-plane.

4. Analysis of the marginal stability curves: 1/2-BPS decay products

4.1 Equations of the curves

To analyse the equation of marginal stability we have obtained, let us first consider the special case when both decay products are $\frac{1}{2}$-BPS. This requires that the electric and magnetic charge vectors of the decay products be proportional. The equation for the charges of the decay products eq. (3.1) can now be written:

$$\begin{pmatrix} m\vec{Q} \\ n\vec{P} \end{pmatrix} \rightarrow \begin{pmatrix} m_1\vec{Q} + r_1\vec{P} \\ s_1\vec{Q} + n_1\vec{P} \end{pmatrix} + \begin{pmatrix} m_2\vec{Q} + r_2\vec{P} \\ s_2\vec{Q} + n_2\vec{P} \end{pmatrix} \tag{4.1}$$

with m_i, n_i, r_i, s_i satisfying:

$$m_1 + m_2 = m, \quad n_1 + n_2 = n, \quad r_1 + r_2 = s_1 + s_2 = 0 \tag{4.2}$$

and where the electric and magnetic (upper and lower) components of each charge vector are proportional to each other. Note that this is the equation for the full, rather than projected, charge vector. The absence of any term out of the plane of $\vec{Q}, \vec{P}$ comes from the fact that if such a term were present, it would be impossible to make the electric and magnetic charges proportional in *both* decay products. Because the above equation is for the full charge vectors, integrality of the charge lattice requires that m_i, r_i, s_i, n_i are integers. In case all four integers (for each i) have a common factor then the decay will be into three or more final states. Since we want to focus on two-body decays, we should exclude such cases.

Proportionality of electric and magnetic charges is equivalent to requiring that the determinant of the associated matrices vanish:

$$\det \begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix} = 0 \tag{4.3}$$

and

$$\det \begin{pmatrix} m - m_1 & -r_1 \\ -s_1 & n - n_1 \end{pmatrix} = 0 \tag{4.4}$$

The first of these equations is solved by the substitution:

$$\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix} = \begin{pmatrix} ad & -ab \\ cd & -bc \end{pmatrix} \tag{4.5}$$

where a, b, c, d are defined only upto an overall reversal of sign. The second equation then tells us that

$$mn + bc\,m - ad\,n = 0 \tag{4.6}$$

Suppose now that the original dyon was primitive, namely $(m, n) = (1, 1)$. In this case eq. (4.6) becomes

$$ad - bc = 1 \tag{4.7}$$

and therefore the decay products are parametrised by a matrix in $PSL(2, Z)$. In going to the coefficients a, b, c, d, we see that they are invariant under the scaling $a, b, c, d \to \lambda a, \lambda^{-1} b, \lambda c, \lambda^{-1} d$ as well as an exchange $a, b, c, d \to -b, a, -d, c$. These transformations, along with eq. (4.7) can be used to show that a, b, c, d are unique integers [9].

Making the substitutions $(m, n) = (1, 1)$ as well as eq. (4.5) in the curve of marginal stability eq. (3.9), and using the $PSL(2, Z)$ property, we find that the curve reduces to:

$$\left(\tau_1 - \frac{ad + bc}{2cd}\right)^2 + \left(\tau_2 + \frac{E}{2cd}\right)^2 = \frac{1}{4c^2d^2}\left(1 + E^2\right) \tag{4.8}$$

where

$$E \equiv \frac{1}{\sqrt{\Delta}}\left(cd\, Q^2 + ab\, P^2 - (ad + bc)Q \cdot P\right) \tag{4.9}$$

This is the equation found by Sen in ref. [9].

These curves are circles with centre at $(\frac{ad+bc}{2cd}, -\frac{E}{2cd})$ and radius $\frac{\sqrt{1+E^2}}{2cd}$. They intersect the real axis in the pair of points

$$\frac{b}{d}, \frac{a}{c} \tag{4.10}$$

Sen showed that, for primitive dyons, two different curves never intersect in the upper half plane but can touch on the real axis in τ-space. This implies that a given primitive $\frac{1}{4}$-BPS dyon can at most be marginally unstable to decay into a single definite pair of $\frac{1}{2}$-BPS dyons at a given point in moduli space.

While the fractions $\frac{b}{d}, \frac{a}{c}$ need not in general be positive or lie between 0 and 1, they can be brought into the form of positive fractions between 0 and 1 by a modular transformation. Suppose for example that $\frac{b}{d}$ does not lie between 0 and 1. Then for some suitable integer N, we define $b' = b - dN$ such that $0 < b' \leq d$. For the same N) we can show that $a' = a - cN$ satisfies $0 < a' \leq c$. As a result, $0 < \frac{a'}{c}, \frac{b'}{d} \leq 1$. The formula for E above is unchanged under this transformation if we simultaneously re-define $\vec{Q} \to \vec{Q} - N\vec{P}$, and the curve of marginal stability is invariant if we also send $\tau_1 \to \tau_1 + N$.

To complete the discussion of decays into $\frac{1}{2}$-BPS final states, we need to consider the case of dyons that are non-primitive, i.e. $(m, n) \neq (1, 1)$. In this case we can obtain the curve of marginal stability by starting from eq. (3.9) and making the appropriate substitutions from eq. (4.5) and eq. (4.6). The coefficients ad, ab, cd, bc are still integers but they no longer describe a matrix in $PSL(2, Z)$. Instead they satisfy the condition:

$$ad\, n - bc\, m = mn \tag{4.11}$$

Moreover, one can check that a, b, c, d are not unique in this case. However only the combinations ad, ab, cd, bc actually appear in the curve of marginal stability, so this curve is unique and can be written:

$$\left(\tau_1 - \frac{nad + mbc}{2ncd}\right)^2 + \left(\tau_2 + \frac{E}{2ncd}\right)^2 = \frac{1}{4n^2c^2d^2}\left(m^2n^2 + E^2\right) \tag{4.12}$$

where

$$E \equiv \frac{1}{\sqrt{\Delta}}\left(mcd\, Q^2 + nab\, P^2 - (nad + mbc)Q \cdot P\right) \tag{4.13}$$

This is the most general curve of marginal stability for decay into $\frac{1}{2}$-BPS dyons.

Examining the curve we find that it intersects the real axis at the points $\frac{a}{c}$ and $\frac{mb}{nd}$. Even though m, n are co-prime, we cannot be sure that mb, nd are co-prime, so the latter fraction is not necessarily reduced to lowest terms. We will discuss the geometry of these curves in a later subsection.

4.2 Farey fractions and Ford circles

In this subsection we briefly review some mathematical constructions that will facilitate the analysis of the $\frac{1}{2}$-BPS curves of marginal stability. In the mathematical literature one encounters the notion of a Farey sequence F_n (see for example ref. [23]). This is the set of all fractions (reduced to lowest terms) with denominators $\leq n$ and taking values in the interval between 0 and 1, arranged in order of increasing magnitude. As an example we have:

$$F_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\} \tag{4.14}$$

Relevant properties of Farey sequences, for us, are the following (more details can be found in ref. [23]). For any pair of fractions $\frac{b}{d}$ and $\frac{a}{c}$ that appear consecutively in *any* Farey sequence, we have $ad - bc = \pm 1$. We can always order them so that the sign is positive, therefore $ad - bc = 1$. Given any such pair, a new fraction called the *mediant* is given by:

$$\text{mediant}\left(\frac{b}{d}, \frac{a}{c} \right) \equiv \frac{a + b}{c + d} \tag{4.15}$$

The mediant lies between the two members of the original pair and will occur between them in subsequent Farey sequences. Moreover, if we define

$$\frac{h}{k} = \frac{a + b}{c + d} \tag{4.16}$$

then $hd - kb = 1 = ak - ch$. Thus the fraction $\frac{h}{k}$ will occur after $\frac{b}{d}$ as well as before $\frac{a}{c}$ in some Farey sequence.

The above construction, which is seen to be related to the structure of the discrete group $\mathrm{PSL}(2, Z)$, can be geometrically visualised in terms of circles called *Ford circles*. These will turn out to be helpful in understanding the properties of the Sen circles of eq. (4.8). For a pair of co-prime integers a, c such that $0 \leq \frac{a}{c} \leq 1$, the associated Ford circle [23] $C\left(\frac{a}{c}\right)$ is a circle centred at $(\frac{a}{c}, \frac{1}{2c^2})$ with radius $\frac{1}{2c^2}$. It is tangent to the horizontal axis at $\frac{a}{c}$, and can be thought of as "sitting above" this fraction. The size of a Ford circle is inversely proportional to the square of the denominator of the fraction. Accordingly the largest possible Ford circles, above the points $\frac{0}{1}$ and $\frac{1}{1}$, have radius $\frac{1}{2}$.

The key property of Ford circles is that (i) two Ford circles never intersect, (ii) two Ford circles associated to the fractions $\frac{b}{d}$ and $\frac{a}{c}$ (without loss of generality we assume the second fraction to be the larger one) are tangent to each other if and only if $ad - bc = 1$. In terms of Farey sequences, if two fractions are consecutive in any Farey sequence then they are associated to a pair of touching Ford circles. Conversely if two Ford circles touch then their corresponding fractions are consecutive in some Farey sequence.
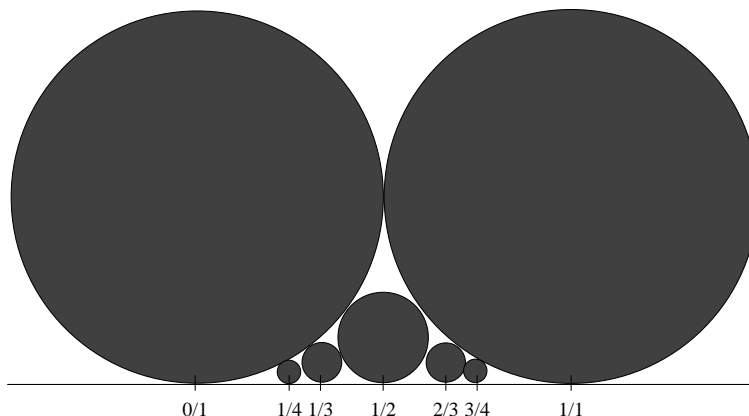
Figure 1: The Ford circles associated to F_4

Finally we describe a construction that will be closely related to marginal decays of dyons. For any pair of touching Ford circles associated to fractions $\frac{b}{d}$ and $\frac{a}{c}$ with $ad-bc = 1$, there is another circle that (for lack of a better name) we will refer to as the "dual Ford circle" $\tilde{C}(\frac{b}{d}, \frac{a}{c})$ that is centred on the real axis and passes through the points $\frac{b}{d}$ and $\frac{a}{c}$ on the real axis. This circle has the property that it also passes through the point at which the two Ford circles touch [23].

4.3 Analysis of the decays: Sen circles and Ford circles

Now let us return to the decay of a primitive $\frac{1}{4}$-BPS dyon into two $\frac{1}{2}$-BPS dyons. As we have seen in the previous subsection, the decay products are defined in terms of a matrix in $\text{PSL}(2, Z)$. This matrix defines a pair of fractions $\frac{b}{d}$ and $\frac{a}{c}$ with $ad - bc = 1$. By the shift $\tau_1 \to \tau_1 + N$, as in the discussion below eq. (4.10), we can make both the fractions lie between 0 and 1. Now the Ford circles associated to these two fractions are tangent to each other. The dual Ford circle $\tilde{C}\left(\frac{b}{d}, \frac{a}{c}\right)$ has its origin on the real axis at the midpoint of these two fractions, at $\frac{ad+bc}{2cd}$. Its radius is given by half the distance between these two fractions, namely $\frac{1}{2cd}$. Thus the equation of this dual Ford circle is:

$$\left(\tau_1 - \frac{ad + bc}{2cd}\right)^2 + \tau_2^2 = \frac{1}{4c^2d^2} \tag{4.17}$$

Comparing with eq. (4.8), we see that the dual Ford circle is the limit of the Sen circle for marginal decays of a primitive $\frac{1}{4}$-BPS dyon into two $\frac{1}{2}$-BPS dyons, as $E \to 0$. (Recall that E was defined in eq. (4.9)). Conversely, the Sen circle can be thought of as a deformation of the dual Ford circle with deformation parameter E. For given integers a, b, c, d, both circles are centred at the same value of τ_1 but have their centres vertically displaced from each other. The radius of the Sen circle is such that it intersects the real axis in the *same* pair of points as the dual Ford circle. Note that $\frac{E}{cd}$ can be positive or negative, so the Sen circle can be displaced either downwards or upwards relative to the dual Ford circle.

This similarity is intriguing and may point to a more profound relation between Sen circles and Ford circles that we have not yet uncovered (in particular, it seems plausible that

by deforming the K3 moduli one can set $E \to 0$, which would make the two circles actually coincide). However, already the relation we have exhibited is sufficient to understand a key property of Sen circles, which is that they do not intersect in the upper half plane, but only on the real axis [9].

This can can be seen as follows. Every Sen circle is associated to a dual Ford circle and thereby to a pair of Ford circles. Consider the two Sen circles associated to a, b, c, d and h, p, k, q with $ad - bc = pk - qh = 1$. Clearly we have $\frac{b}{d} < \frac{a}{c}$ as well as $\frac{h}{k} < \frac{p}{q}$. The two possible orderings of the fractions are $\frac{b}{d}, \frac{h}{k}, \frac{a}{c}, \frac{p}{q}$ and $\frac{b}{d}, \frac{a}{c}, \frac{h}{k}, \frac{p}{q}$. The first ordering is ruled out by the Ford circle construction, since it implies that the Ford circle of the first fraction touches that of the third one, while the Ford circle of the second fraction touches that of the fourth one. This contradicts the fact that all the Ford circles are non-overlapping. Thus only the second ordering is possible, where we have the fractions $\frac{b}{d}, \frac{a}{c}, \frac{h}{k}, \frac{p}{q}$ in increasing order. Let us consider the case where $\frac{a}{c} = \frac{h}{k}$, so that the Sen circles touch on the real axis. Clearly the dual Ford circles also touch on the real axis, which means the three fractions $\frac{b}{d}, \frac{a}{c}, \frac{p}{q}$ are consecutive terms in a Farey sequence.

We want to show that the Sen circles in this case do not intersect in the upper half plane. This imposes a condition on the slopes of the Sen circles at the real axis. From eq. (4.8) we find that the slope at the real axis is given by:

$$\tan \phi = \pm \frac{1}{E} \tag{4.18}$$

where the two signs hold for the two intersection points. Now it is easy to check that the condition we are seeking is:

$$E(a, b, c, d) + E(a, p, c, q) > 0 \tag{4.19}$$

This is of course satisfied if both E's are positive, though that is not in general the case. But even in the general case the condition above does hold, as we now demonstrate. From the definition of E one finds that:

$$E(a, b, c, d) + E(a, p, c, q) = \frac{1}{\sqrt{\Delta}} \Big(c(q + d)Q^2 + a(p + b)P^2 - (a(q + d) + c(p + b)) P \cdot Q \Big) \tag{4.20}$$

Now we use the fact, explained in the discussion around eq. (4.16), that if three fractions are consecutive in a Farey sequence then the middle one is the mediant of the other two. Hence we have:

$$\frac{a}{c} = \frac{p + b}{q + d} \tag{4.21}$$

from which we get:

$$Na = (p + b), \quad Nc = (q + d) \tag{4.22}$$

for some integer $N \geq 1$. It follows that:

$$E(a, b, c, d) + E(a, p, c, q) = \frac{N}{\sqrt{\Delta}} (c\vec{Q} - a\vec{P})^2 > 0 \tag{4.23}$$

as desired. By similar methods the non-intersecting property of Sen circles can be proved for the case where $\frac{b}{d}, \frac{a}{c}, \frac{h}{k}, \frac{p}{q}$ are all distinct fractions.

4.4 Analysis of the decays: non-primitive case

The above discussion was for the case of a primitive dyon as the initial state. Now let us look at the case where the initial state is a non-primitive dyon. In this case the Sen circle is replaced by eq. (4.12), which intersects the real axis at the points $\frac{a}{c}$ and $\frac{mb}{nd}$. Let us now analyse the condition eq. (4.11) in some detail. Because m and n are co-prime, writing this condition as $adn = m(bc + n)$ tells us that m divides ad and also that n divides bc. Therefore we can rewrite eq. (4.11) as:

$$\frac{ad}{m} - \frac{bc}{n} = 1 \tag{4.24}$$

where each of the terms on the l.h.s. is an integer. This can only be realised if, for some (not necessarily prime or unique) factorisation of m and n;

$$m = pq, \quad n = kl \tag{4.25}$$

we have that:

$$a' = \frac{a}{p}, \quad b' = \frac{b}{k}, \quad c' = \frac{c}{l}, \quad d' = \frac{d}{q} \tag{4.26}$$

are all integers. Evidently they satisfy $a'd' - b'c' = 1$. Substituting in the curve of marginal stability for this case, eq. (4.12), we find:

$$\left(\tau_1 - \frac{p}{l}\frac{a'd' + b'c'}{2c'd'}\right)^2 + \left(\tau_2 + \frac{p}{l}\frac{E'}{2c'd'}\right)^2 = \frac{p^2}{4l^2 c'^2 d'^2}\left(1 + E'^2\right) \tag{4.27}$$

where

$$E' \equiv \frac{mn}{\sqrt{\Delta}}\left(\frac{q}{k}c'd'\,Q^2 + \frac{k}{q}a'b'\,P^2 - (a'd' + b'c')Q \cdot P\right) \tag{4.28}$$

This curve intersects the real axis at the points:

$$\frac{p}{l}\frac{b'}{d'}, \qquad \frac{p}{l}\frac{a'}{c'}, \tag{4.29}$$

For a fixed value of $\frac{p}{l}$, the set of intersection points is in one to one correspondence with those for the primitive case, where using Ford circles (or the methods of ref. [9]) we saw that curves of marginal stability do not intersect. However the value of $\frac{p}{l}$ is not fixed. For given m, n specifying a non-primitive dyon, eq. (4.25) permits several solutions for p and l in general. For each of them we obtain a construction in 1-1 correspondence with the set of curves of marginal stability for the primitive case, and it appears quite likely that curves from one of these sets can intersect with curves from another set. This would result in curves of marginal stability that intersect each other in the upper half plane.

To generate examples, it is convenient to revert to the notation in which the charges of the decay products are labelled by a matrix of integers $\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix}$ satisfying eqs. (4.3) and (4.4). From these two equations we find that:

$$m_1 n + n_1 m = mn \tag{4.30}$$

– 12 –

from which we see that m_1 is a multiple of m. We write:

$$m_1 = m\alpha_1 \tag{4.31}$$

where α_1 is another integer. The equations now yield the following general form for the matrix:

$$\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix} = \begin{pmatrix} m\alpha_1 & \frac{mn\,\alpha_1(1-\alpha_1)}{s_1} \\ s_1 & n(1-\alpha_1) \end{pmatrix} \tag{4.32}$$

The strategy is now to choose a value for α_1 and then look for the set of s_1 that divide $mn\,\alpha_1(1-\alpha_1)$. Finally to ensure that we are dealing with a two-body decay, we must check that there is no overall common factor in either of the matrices

$$\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix}, \quad \begin{pmatrix} m - m_1 & -r_1 \\ -s_1 & n - n_1 \end{pmatrix} \tag{4.33}$$

In this way we can generate a large number of examples of curves of marginal stability for non-primitive dyons decaying into a pair of $\frac{1}{2}$-BPS dyons.

To check the possible intersections of such curves, we recall that they intersect the real axis in the points $\frac{m_1-m}{s_1}$ and $\frac{m_1}{s_1}$. If two such intervals intersect then the curves will necessarily intersect in the upper half-plane. Let us consider a definite example. Suppose $(m, n) = (2, 3)$. Then choosing $\alpha_1 = 1$, we see that s_1 can be arbitrary. On the other hand choosing $\alpha_1 = 2$ we find that the allowed values of s_1 are $1, 2, 3, 4, 6, 12$. It is easy to check that for the very simplest choices the curves do not intersect. However, picking $\alpha_1 = 1, s_1 = 7$ and $\alpha_1 = 2, s_1 = 12$ we find that all the conditions are satisfied and the decay products are given by the matrices:

$$\alpha_1 = 1, \ s_1 = 7 : \qquad \begin{pmatrix} 2 & 0 \\ 7 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ -7 & 3 \end{pmatrix}$$

$$\alpha_1 = 2, \ s_1 = 12 : \qquad \begin{pmatrix} 4 & -1 \\ 12 & -3 \end{pmatrix}, \quad \begin{pmatrix} -2 & 1 \\ -12 & 6 \end{pmatrix} \tag{4.34}$$

In terms of the integers a, b, c, d the two decay processes are parametrised by the matrices:

$$(i) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 7 & 1 \end{pmatrix}$$

$$(ii) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \tag{4.35}$$

Each matrix satisfies $3ad - 2bc = 6$.

Now the curves of marginal stability for the two decay modes intersect the real axis at the following values:

$$(i) \quad \tau_1 = 0, \frac{2}{7}$$

$$(ii) \quad \tau_1 = \frac{1}{6}, \frac{1}{3} \tag{4.36}$$

These two intervals are overlapping, hence the associated curves must intersect in the interior of the upper half plane. We conclude that curves of marginal stability for the decay of non-primitive dyons can in general intersect in the upper half plane, unlike what happens for primitive dyons. It would be important to understand the physical and mathematical reasons why the curves intersect, as well as the consequences of this fact.

5. Analysis of the marginal stability curves: 1/4-BPS decay products

5.1 Decays into a $1/2$-BPS and a $1/4$-BPS dyon

We now consider decays of a $\frac{1}{4}$-BPS dyon into one $\frac{1}{2}$-BPS and one $\frac{1}{4}$-BPS dyon. This is parametrised as in eq. (3.2). If the first decay product is taken to be $\frac{1}{2}$-BPS then we must impose the condition eq. (4.3) which is solved by eq. (4.5). However, the coefficients m_i, r_i, s_i, n_i are no longer required to be integers and therefore nor are a, b, c, d. Moreover we want the second state to be $\frac{1}{4}$-BPS and therefore $adn - bcm \neq mn$. Finally, as indicated earlier, we must check that the curve we obtain from eq. (3.9) actually describes the forward and not the reverse decay process.

Consider the case where the initial state is a primitive dyon with $(m, n) = (1, 1)$. For this case we find the curve of marginal stability to be:

$$\left(\tau_1 - \frac{m_1 - n_1}{2s_1}\right)^2 + \left(\tau_2 + \frac{E}{2s_1}\right)^2 = \frac{1}{4s_1^2}\Big((m_1 - n_1)^2 + 4r_1 s_1 + E^2\Big) \tag{5.1}$$

where

$$E \equiv \frac{1}{\sqrt{\Delta}}\left(s_1 Q_R^2 - r_1 P_R^2 - (m_1 - n_1)Q_R \cdot P_R\right) \tag{5.2}$$

On replacing m_1, r_1, s_1, n_1 by their expressions in terms of a, b, c, d we can also bring it to the form:

$$\left(\tau_1 - \frac{ad + bc}{2cd}\right)^2 + \left(\tau_2 + \frac{E}{2cd}\right)^2 = \frac{1}{4c^2 d^2}\left((ad - bc)^2 + E^2\right) \tag{5.3}$$

with

$$E \equiv \frac{1}{\sqrt{\Delta}}\left(cd\, Q_R^2 + ab\, P_R^2 - (ad + bc)Q_R \cdot P_R\right) \tag{5.4}$$

The equation is very similar to the Sen circle for decays of a primitive dyon into $\frac{1}{2}$-BPS decay products. However, the constraints on a, b, c, d are quite different. Instead of analysing this case further, we will return to it as a special case of the more general decay into two $\frac{1}{4}$-BPS states.

5.2 Decays into two $1/4$-BPS dyons

We now address the case in which the initial $\frac{1}{4}$-BPS dyon decays into a pair of $\frac{1}{4}$-BPS dyons. Again we start with the primitive case, $(m, n) = (1, 1)$. The relevant curve of marginal stability is the same as in the previous subsection, eq. (5.1), except that the determinants of $\begin{pmatrix} m_i & r_i \\ s_i & n_i \end{pmatrix}$ are both nonzero. (Later we will also be able to specialise to the case where one of them is zero.)

Let us now address the constraints on the final state parameters that are required to ensure that the decay process corresponds to the correct branch of eq. (3.6). First of all, the quantity Δ that appears in the BPS mass formula eq. (2.10) involves a square root, and we have taken all square roots to be positive. This has the following consequence. Observe that:

$$\Delta(m_i\vec{Q} + r_i\vec{P}, s_i\vec{Q} + n_i\vec{P}) = \det \begin{pmatrix} m_i & r_i \\ s_i & n_i \end{pmatrix} \Delta(\vec{Q}, \vec{P}) \tag{5.5}$$

Positivity of Δ on both sides of the equation imposes the condition:

$$\det \begin{pmatrix} m_i & r_i \\ s_i & n_i \end{pmatrix} > 0, \qquad i = 1, 2 \tag{5.6}$$

Since

$$\begin{pmatrix} m_2 & r_2 \\ s_2 & n_2 \end{pmatrix} = \begin{pmatrix} 1 - m_1 & -r_1 \\ -s_1 & 1 - n_1 \end{pmatrix} \tag{5.7}$$

we find that:

$$m_1 n_1 - r_1 s_1 > \max(m_1 + n_1 - 1, 0) \tag{5.8}$$

For what follows, it will be convenient to introduce the eigenvalues β_1, γ_1 of $\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix}$ and β_2, γ_2 of $\begin{pmatrix} m_2 & r_2 \\ s_2 & n_2 \end{pmatrix}$. Because the two matrices commute (they are of the form $\mathbf{F}$ and $1 - \mathbf{F}$) they can be simultaneously diagonalised, from which we see that:

$$\beta_1 + \beta_2 = 1 = \gamma_1 + \gamma_2 \tag{5.9}$$

From the determinant conditions above, we have:

$$\beta_1 \gamma_1 > 0, \quad (1 - \beta_1)(1 - \gamma_1) > 0 \tag{5.10}$$

We will now examine the quantities $\frac{M_1}{M}, \frac{M_2}{M}$ on the curve eq. (5.1). For convenience, we would like to choose a particular point on the curve and evaluate these quantities there. The possible results are as follows. If we find $\frac{M_1}{M} > 1$ at a point, then the marginal stability curve cannot correspond to $M = M_1 + M_2$. It may correspond to either $M_1 = M + M_2$ or $M_2 = M + M_1$. Which of the two cases it corresponds to is then not very important, but can be distinguished by looking at $\frac{M_2}{M}$. If on the other hand we find $\frac{M_1}{M} < 1$ then we have the possibilities of being on the correct branch $M = M_1 + M_2$ or on the wrong branch $M_2 = M + M_1$. This time it is essential to distinguish the two, which can again be done by evaluating $\frac{M_2}{M}$. Being on the correct branch requires $\frac{M_i}{M} < 1$ for both $i = 1$ and 2.

In any of these cases, having determined the relevant branch of eq. (3.6) at one point on the curve, we can be sure that we will not cross over to another branch elsewhere on the same curve, since crossing from one branch to another requires passing through a point where one of the masses vanishes. But the BPS mass formula does not vanish for any value of the moduli, so this is not possible (unless the charges of that state vanish identically).

Let us assume that the matrix $\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix}$ is such that the curve eq. (5.1) intersects the real axis. This will happen if the eigenvalues β_1, γ_1 are both real (without loss of generality we take $\gamma_1 \geq \beta_1$). Then, a convenient point at which to evaluate the mass ratios is one of the intersection points of the curve with the real axis. Setting $\tau_2 = 0$ in eq. (3.9), we get the following equation for τ_1:

$$n_1 - \frac{r_1}{\tau_1} = m_1 - \tau_1 s_1 \tag{5.11}$$

Now let us consider the expression $\frac{M_1^2}{M^2}$ in the limit $\tau_2 \to 0$. We have:

$$\begin{aligned} \left.\frac{M_1^2}{M^2}\right|_{\tau_2 \to 0} &= \frac{[(m_1 \vec{Q}_R + r_1 \vec{P}_R) - \tau_1(s_1 \vec{Q}_R + n_1 \vec{P}_R)]^2}{[\vec{Q}_R - \tau_1 \vec{P}_R]^2} \\ &= \frac{[(m_1 - \tau_1 s_1)\vec{Q}_R - \tau_1(n_1 - \frac{r_1}{\tau_1})\vec{P}_R]^2}{[\vec{Q}_R - \tau_1 \vec{P}_R]^2} \end{aligned} \tag{5.12}$$

Using eq. (5.11) we now get:

$$\left.\frac{M_1}{M}\right|_{\tau_2 \to 0} = |m_1 - \tau_1 s_1| \tag{5.13}$$

On the real axis, eq. (5.1) gives:

$$\tau_1 = \frac{1}{2s_1}\Big(\pm (\gamma_1 - \beta_1)| + (m_1 - n_1) \Big) \tag{5.14}$$

Inserting this into eq. (5.13) we find:

$$m_1 - \tau_1 s_1 = \beta_1 \text{ or } \gamma_1 \tag{5.15}$$

Let us first consider the case $m_1 n_1 - r_1 s_1 > 1$. We will show that in this region the decay is not the desired one, but corresponds instead to a branch of eq. (3.6) describing a reverse decay. With this condition on the determinant, one of the eigenvalues (say γ_1) must be > 1. Positivity of the second determinant, which equals $(1 - \beta_1)(1 - \gamma_1)$, tells us that if $\gamma_1 > 1$ then also $\beta_1 > 1$. Thus we have that both eigenvalues are > 1. It follows that $\frac{M_1}{M} > 1$ and we are, as promised, on the wrong branch.

Next suppose $m_1 n_1 - r_1 s_1 = 1$. The above considerations then show that $\beta_1 = \gamma_1 = 1$. Then we $\frac{M_1}{M} = 1$. This means $M_2 = 0$ and therefore the charges associated to the second state are identically zero. In other words, $\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. This is a trivial case where the first decay product is the original state itself.

Let us note at this point that if m_1, r_1, s_1, n_1 had been taken to be integers, and the corresponding state was restricted not to be $\frac{1}{2}$-BPS, we would necessarily have $m_1 n_1 - r_1 s_1 \geq 1$. We have shown that all such cases do not correspond to a valid decay of M into M_1 and M_2, therefore no such decays exist for integer coefficients. This is one of the key results of ref. [14].

That leaves the case

$$0 < m_1 n_1 - r_1 s_1 < 1, \quad 0 < m_2 n_2 - r_2 s_2 < 1 \tag{5.16}$$

which can only be satisfied for fractional coefficients.

Requiring $\beta_1\gamma_1 < 1$ and also $\beta_2\gamma_2 = (1 - \beta_1)(1 - \gamma_1) < 1$ we see that $0 < \beta_1, \gamma_1 < 1$ and $0 < \beta_2, \gamma_2 < 1$. From this and eq. (5.15) we find $\frac{M_1}{M} < 1, \frac{M_2}{M} < 1$ and this indeed corresponds to the decay process that we were looking for. Thus eq. (5.16) provides a necessary condition for the coefficients m_1, n_1, r_1, s_1 in eq. (3.2) in order to have a decay of the original dyon into two $\frac{1}{4}$-BPS dyons. Under this condition, our curve eq. (3.9) describes the marginal stability locus in the τ_1, τ_2 plane. However this is a locus of co-dimension 2 in the full moduli space, for the following reason. Fractional m_1, r_1, s_1, n_1 means that the decay process in terms of the original integral charge vectors was into states living outside the $\vec{Q}, \vec{P}$ plane. This is precisely the case, referred to earlier, where the moduli in M need to be adjusted to make the final state charges (after R projection) lie in the same plane as the initial state charges [14]. It remains to find a sufficient condition on the values of m_1, r_1, s_1, n_1 as well as to understand more precisely the condition on the moduli matrix M which put the projected charge vectors in the plane of the decaying dyon.

6. Discussion

We have found a general equation for marginal stability of $\frac{1}{4}$-BPS dyons to decay into two final state particles, eq. (3.9). Analysis of the equation reveals many distinct cases with different properties. We believe this analysis can be easily extended to multi-particle final states. The construction of Ford circles and especially their dual circles proved useful in this analysis and we suspect that there may be a deeper mathematical relationship to the Sen circles of marginal instability for primitive dyons decaying into $\frac{1}{2}$-BPS final states.

To complete the discussion of the previous section of decays into $\frac{1}{4}$-BPS states we need to consider the case where the eigenvalues of $\begin{pmatrix} m_1 & r_1 \\ s_1 & n_1 \end{pmatrix}$ are complex, in which case the curve of marginal stability does not intersect the real axis. Also we need to generalise to the case of non-primitive initial states.

We see that decays into $\frac{1}{2}$-BPS final states are labelled by integers a, b, c, d that are in $\mathrm{PSL}(2, Z)$ for primitive initial states and obey the more complicated relation eq. (4.6) when the initial state is a non-primitive dyon. However, such integers do not occur when the final state consists of $\frac{1}{4}$-BPS dyons. It would be nice to understand the physical origin of this $\mathrm{PSL}(2, Z)$ and its generalisations in the cases where they occur. We expect this can be done through the string network [20 – 22] representation of the dyons.

The role of non-primitive dyons and their decays has been somewhat mysterious[3] since the observation of ref. [10] that the quantum states of such dyons are not counted by the famous genus-2 modular form of ref. [15] but appear to be connected to a higher-genus Riemann surface. We have exhibited the $\frac{1}{2}$-BPS decays of such dyons and noted that their curves of marginal stability intersect in the interior of moduli space. This may be helpful in resolving the puzzle of their role in the counting problem.

---

[3]and needs to be taken up by the Department of Mysteries, Ministry of Magic.

Acknowledgments

References

[1] D. Shih, A. Strominger and X. Yin, *Recounting dyons in N = 4 string theory*, *JHEP* **10** (2006) 087 [hep-th/0505094].

[2] D. Gaiotto, *Re-recounting dyons in N = 4 string theory*, hep-th/0506249.

[3] D.P. Jatkar and A. Sen, *Dyon spectrum in CHL models*, *JHEP* **04** (2006) 018 [hep-th/0510147].

[4] J.R. David, D.P. Jatkar and A. Sen, *Product representation of dyon partition function in CHL models*, *JHEP* **06** (2006) 064 [hep-th/0602254].

[5] J.R. David and A. Sen, *CHL dyons and statistical entropy function from D1 − D5 system*, *JHEP* **11** (2006) 072 [hep-th/0605210].

[6] J.R. David, D.P. Jatkar and A. Sen, *Dyon spectrum in N = 4 supersymmetric type-II string theories*, *JHEP* **11** (2006) 073 [hep-th/0607155].

[7] J.R. David, D.P. Jatkar and A. Sen, *Dyon spectrum in generic N = 4 supersymmetric Z_N orbifolds*, *JHEP* **01** (2007) 016 [hep-th/0609109].

[8] A. Dabholkar and D. Gaiotto, *Spectrum of CHL dyons from genus-two partition function*, hep-th/0612011.

[9] A. Sen, *Walls of marginal stability and dyon spectrum in N = 4 supersymmetric string theories*, *JHEP* **05** (2007) 039 [hep-th/0702141].

[10] A. Dabholkar, D. Gaiotto and S. Nampuri, *Comments on the spectrum of CHL dyons*, hep-th/0702150.

[11] N. Banerjee, D.P. Jatkar and A. Sen, *Adding charges to N = 4 dyons*, *JHEP* **07** (2007) 024 [arXiv:0705.1433].

[12] A. Sen, *Two centered black holes and N = 4 dyon spectrum*, arXiv:0705.3874.

[13] M.C.N. Cheng and E. Verlinde, *Dying dyons don't count*, arXiv:0706.2363.

[14] A. Sen, *Rare decay modes of quarter BPS dyons*, arXiv:0707.1563.

[15] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, *Counting dyons in N = 4 string theory*, *Nucl. Phys.* **B 484** (1997) 543 [hep-th/9607026].

[16] F. Denef and G.W. Moore, *Split states, entropy enigmas, holes and halos*, hep-th/0702146.

[17] M. Cvetič and D. Youm, *Dyonic BPS saturated black holes of heterotic string on a six torus*, *Phys. Rev.* **D 53** (1996) 584 [hep-th/9507090].

[18] M.J. Duff, J.T. Liu and J. Rahmfeld, *Four-dimensional string-string-string triality*, *Nucl. Phys.* **B 459** (1996) 125 [`hep-th/9508094`].

[19] S. Lang, *Algebra*, Addison-Wesley (1993).

[20] O. Aharony, A. Hanany and B. Kol, *Webs of (p,q) 5-branes, five dimensional field theories and grid diagrams*, *JHEP* **01** (1998) 002 [`hep-th/9710116`].

[21] K. Dasgupta and S. Mukhi, *BPS nature of 3-string junctions*, *Phys. Lett.* **B 423** (1998) 261 [`hep-th/9711094`].

[22] A. Sen, *String network*, *JHEP* **03** (1998) 005 [`hep-th/9711130`].

[23] T. Apostol, *Modular functions and Dirichlet series in number theory*, Springer-Verlag (1976).